Koopman Operator Theory for Computational Neuroscientists

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1 Introduction

Takeaway. Linear systems are understood *globally* through their spectrum. Koopman lifts nonlinear dynamics to *observables* to recover that same global linear picture.

Goal: understand how a state $x \in \mathcal{X}$ evolves.

- Continuous time (vector field f): $\dot{x} = f(x)$.
- Discrete time (map g): $x_{t+1} = g(x_t)$.

Linear systems → global. For $\dot{x} = Ax$, diagonalize $A = V\Lambda V^{-1}$ into decoupled modes:

$$\dot{v}_i = \lambda_i v_i,$$
$$v_i(t) = e^{\lambda_i t}\, v_i(0).$$

- $\operatorname{Re}\lambda_i$: growth/decay. $\operatorname{Im}\lambda_i$: oscillation. Eigenvectors: simple directions.
- Spectrum = complete, *global* description of all of state space.

Nonlinear → only local. Linearize at a fixed point x^* ($f(x^*) = 0$):

$$J = \frac{\partial f}{\partial x}\bigg|_{x^*},$$
$$J = V\Lambda V^{-1}.$$

- Hartman–Grobman: valid *only* near x^*.
- Silent in the transient / input-driven / multi-attractor regimes where neural computation lives.

Koopman idea. Track *observables* $\phi(x)$ instead of the state x.

- Observables evolve under a *linear* operator U — even when f, g are nonlinear.
- Price: U acts on an infinite-dimensional function space. Reward: eigenvalues/spectra return, globally.
- A Koopman eigenfunction ψ evolves as a single exponential, $\dfrac{d}{dt}\psi(x) = \lambda\,\psi(x)$ — the nonlinear analogue of a linear eigenmode.

2 Preliminaries

Takeaway. Observables are measurements in a Hilbert space; the flow map is what turns continuous time into discrete time.

- State space $\mathcal{X}$ (usually $\mathbb{R}^n$).
- Vector field f: $\dot{x} = f(x)$. Map g: $x_{t+1} = g(x_t)$, $t \in \mathbb{Z}$. (Keep these distinct.)

Flow map F^t (continuous → discrete). With $x(t)$ solving $\dot{x} = f(x)$ from x_0:

$$\begin{aligned} F^t(x_0) &= x(t), \\ \tfrac{d}{dt} F^t(x) &= f\big(F^t(x)\big), \\ F^0 &= \mathrm{id}, \\ F^{t+s} &= F^t \circ F^s. \end{aligned}$$

- Sample at timestep τ: $g = F^\tau$. That is how a flow becomes a map (shared spectra, up to $\lambda \mapsto e^{\lambda\tau}$).

Observables & their Hilbert space.

- Observable: scalar $\phi : \mathcal{X} \to \mathbb{C}$ (a measurement of the state).
- Live in a Hilbert space $\mathcal{H}$ (∞-dim Euclidean space + inner product). Canonical choice:

$$\begin{aligned} \mathcal{H} &= L^2(\mu) \\ &= \Big\{\phi \;\Big|\; \textstyle\int_{\mathcal{X}} |\phi(x)|^2 d\mu < \infty\Big\}, \\ \langle \phi_1, \phi_2 \rangle &= \textstyle\int_{\mathcal{X}} \overline{\phi_1(x)}\, \phi_2(x)\, d\mu. \end{aligned}$$

- The map $x \mapsto \phi$ is the "lift": finite-dim nonlinear → infinite-dim linear.
- A linear operator U on $\mathcal{H}$ obeys $U(\alpha x + \beta y) = \alpha U x + \beta U y$, and is *bounded* if $\|Ux\| \le c\|x\|$ for some finite c.
- Finite dims: always bounded. Infinite dims: can fail ⇒ *continuous spectrum* (tricky for estimation).

Notation (fixed throughout).

f	vector field, $\dot{x} = f(x)$	g: map, $x_{t+1} = g(x_t)$
F^t	flow map of f ($g = F^\tau$)	ϕ: observable; $\vec{\phi}$: vector observable
U	Koopman operator	$\mathcal{L}$: generator
ψ	eigenfunction ($U\psi = \lambda\psi$)	λ: eigenvalue
v	Koopman mode	ρ: density; $\mathcal{P}$: Perron–Frobenius operator

3 The Koopman Operator

Takeaway. U evolves observables by *composition* with the dynamics — and it is linear even when the dynamics are not.

- Definition (discrete), evolving an observable = evaluate it *after* one step of g:

$$\begin{aligned} U\phi &= \phi \circ g, \\ U(\phi(x)) &= \phi(g(x)). \end{aligned}$$

- Linear (commutes under scalar multiplication and pointwise addition):

$$\begin{aligned} U(\alpha\phi_1 + \beta\phi_2)(x) &= (\alpha\phi_1 + \beta\phi_2)(g(x)) && \text{definition of the Koopman operator} \\ &= \alpha\phi_1(g(x)) + \beta\phi_2(g(x)) && \text{functions add pointwise} \\ &= \alpha U\phi_1(x) + \beta U\phi_2(x). && \text{definition of the Koopman operator} \end{aligned}$$

- Contains the original system (the identity observable is in the space).

- Continuous time: swap g for the flow F^t, so $U^t\phi = \phi \circ F^t$, and U^t inherits the semigroup property:

$$U^{t+s} = U^t U^s,$$
$$U^0 = I.$$

- Generator (the "vector field" of U), via the chain rule:

$$\begin{aligned}
\mathcal{L}\phi &= \lim_{t\to 0} \frac{U^t\phi - \phi}{t} && \text{definition of the generator} \\
&= \frac{d}{dt}\Big|_{t=0} (\phi \circ F^t) && \text{the semigroup acts by composition} \\
&= \frac{d}{dt}\phi\big(x(t)\big) && \text{evaluate along the trajectory} \\
&= \nabla\phi \cdot \dot{x} && \text{chain rule} \\
&= \nabla\phi \cdot f && \text{substitute the dynamics (Koopman PDE)}
\end{aligned}$$

- Stochastic systems (take the expectation over the dynamics):

$$U\phi(x) = \mathbb{E}[\phi \circ g(X) \mid X = x],$$
$$\mathcal{L}\phi = \lim_{dt\to 0} \frac{\mathbb{E}[d\phi]}{dt}.$$

Dual: the Perron–Frobenius operator

Takeaway. U pushes observables forward; $\mathcal{P}$ pushes *densities* forward. They are adjoints, so their spectra are complex conjugates.

- $\mathcal{P}$ evolves densities ρ — mass into A equals mass from its preimage:

$$\int_A (\mathcal{P}\rho)(x)\,dx = \int_{g^{-1}(A)} \rho(x)\,dx.$$

- Adjoint under $\langle \phi, \rho\rangle = \int \phi\rho$ (finite-dim analogue $\langle Av, w\rangle = \langle v, A^\top w\rangle$): $\langle U\phi, \rho\rangle = \langle \phi, \mathcal{P}\rho\rangle$. Test on an indicator $\phi = \mathbf{1}_A$ (with $U\mathbf{1}_A = \mathbf{1}_{g^{-1}(A)}$):

$$\begin{aligned}
\langle U\mathbf{1}_A, \rho\rangle &= \int_{g^{-1}(A)} \rho\,dx && \text{the indicator pulls back to its preimage} \\
&= \int_A \mathcal{P}\rho\,dx && \text{the defining mass identity} \\
&= \langle \mathbf{1}_A, \mathcal{P}\rho\rangle && \text{definition of the pairing}
\end{aligned}$$

- Linear in ϕ, and indicators span the observables, so it holds for every ϕ.
- Hence conjugate spectra:

$$\begin{aligned}
\sigma(\mathcal{P}) &= \sigma(U^*) && \text{Perron–Frobenius is the adjoint} \\
&= \overline{\sigma(U)} && \text{the adjoint conjugates the spectrum} \\
&= \{\bar{\lambda} : \lambda \in \sigma(U)\}. && \text{write the conjugate set out}
\end{aligned}$$

- Same eigenvalue magnitudes; every real eigenvalue is shared.

→ *With U linear, we can do spectral theory.*

4 Spectral Theory

Takeaway. Koopman turns nonlinear dynamics into a spectrum: eigenvalue = time behavior, eigenfunction = projection, Koopman mode = direction.

- Eigen-equation: $U\psi = \lambda\psi$, i.e. $(U\psi)(x) = \lambda\psi(x)$.
- Point spectrum (null space of $U - \lambda I$) vs. continuous spectrum (resolvent $(U - \lambda I)^{-1}$ unbounded; appears for chaos/mixing). We focus on the point spectrum.
- The triple (λ, ψ, v): eigenvalue, eigenfunction, Koopman mode.

Koopman mode decomposition. For a vector observable $\vec{\phi} : \mathcal{X} \to \mathbb{C}^m$ (with $v_k = \langle \vec{\phi}, \psi_k \rangle$ if the ψ_k are orthonormal):

$$\begin{aligned} \vec{\phi}(x) &= \sum_k \psi_k(x)\, v_k, \\ (U^t \vec{\phi})(x) &= \sum_k \lambda_k^t\, \psi_k(x)\, v_k, \\ x_t &= \sum_k \lambda_k^t\, \psi_k(x_0)\, v_k. \end{aligned}$$

- Match to $A = V\Lambda V^{-1}$ acting as $Ax = V\Lambda V^{-1}x$: modes $= V$, eigenvalues $= \Lambda$, eigenfunctions $= V^{-1}x$.
- Discrete $\leftrightarrow$ continuous: if $\mathcal{L}\psi = \lambda_{\mathcal{L}}\psi$ then $U^\tau \psi = e^{\tau\lambda_{\mathcal{L}}}\psi$, so $\lambda_U = e^{\tau\lambda_{\mathcal{L}}}$. Eigenvalues *multiply* (discrete), *add* (continuous).
- Along a trajectory, an eigenfunction is a single exponential:

$$\begin{aligned} \psi(x(t)) &= e^{\lambda t}\, \psi(x(0)), \\ |\psi(x(t))| &= e^{\mathrm{Re}\,\lambda\, t}\, |\psi(x(0))|. \end{aligned}$$

- Constant $\psi(x) = 1$ is always an eigenfunction: $\lambda = 1$ (discrete), 0 (continuous).

Algebra of eigenfunctions (a multiplicative lattice / abelian semigroup). Products of eigenfunctions are eigenfunctions. Discrete (composition splits over products): with $(\psi_1\psi_2)(x) = \psi_1(x)\psi_2(x)$

$$\begin{aligned} U(\psi_1\psi_2) &= \psi_1\psi_2 \circ g && \text{definition of the Koopman operator} \\ &= (\psi_1 \circ g)(\psi_2 \circ g) && \text{composition distributes over the product} \\ &= (U\psi_1)(U\psi_2) && \text{definition of the operator on each factor} \\ &= \lambda_1\lambda_2(\psi_1\psi_2) && \text{the eigenfunction equations} \end{aligned}$$

Continuous (product rule on $\mathcal{L} = \nabla(\cdot) \cdot f$):

$$\begin{aligned} \mathcal{L}(\psi_1\psi_2) &= \nabla(\psi_1\psi_2) \cdot f && \text{definition of the generator} \\ &= \psi_1(\nabla\psi_2 \cdot f) + (\nabla\psi_1 \cdot f)\psi_2 && \text{product rule} \\ &= \psi_1\mathcal{L}\psi_2 + (\mathcal{L}\psi_1)\psi_2 && \text{definition of the generator} \\ &= (\lambda_1 + \lambda_2)(\psi_1\psi_2) && \text{the eigenfunction equations} \end{aligned}$$

- $\Rightarrow$ powers λ^n are eigenvalues; a few *generating* eigenvalues produce the rest.
- *Linear system:* Koopman eigenvalues = the linear eigenvalues, plus this lattice.

$\to$ *The spectrum provides interpretable structure of the dynamics that correspond to desirable properties, as we'll see next.*

5 Useful properties of Koopman spectra

Closed (invariant) Koopman subspaces

Takeaway. A finite set of eigenfunctions gives an *exact* finite linear model of the nonlinear dynamics.

- Closed subspace $\mathcal{S} \subseteq \mathcal{H}$: $U\mathcal{S} \subseteq \mathcal{S}$, i.e. $\phi \in \mathcal{S} \Rightarrow U\phi \in \mathcal{S}$ (the invariant-subspace idea, lifted to observables).
- Span of eigenfunctions is closed; on it U acts as $\mathrm{diag}(\lambda_1, \ldots, \lambda_p)$:

$$U \sum_{i=1}^{p} c_i \psi_i = \sum_{i=1}^{p} c_i \lambda_i \psi_i \qquad \text{(nonlinear dynamics represented \textit{exactly} by a finite matrix).}$$

- Products of eigenfunctions are eigenfunctions, so they give an endless family of closed subspaces:

$$U\big(\psi_1^{k_1} \cdots \psi_p^{k_p}\big) = \big(\lambda_1^{k_1} \cdots \lambda_p^{k_p}\big)\, \psi_1^{k_1} \cdots \psi_p^{k_p}.$$

- *Linear example:* $x_{t+1} = Ax_t$, A diagonalizable. Eigenfunctions are $\psi_i(x) = w_i^\top x$ (left eigenvectors $w_i^\top A = \lambda_i w_i^\top$), since $U\psi_i(x) = w_i^\top Ax = \lambda_i w_i^\top x = \lambda_i \psi_i(x)$. Monomials up to total degree d form a finite closed subspace — exactly linear.

Worked example: $\dot{x} = x - x^3$ (double well)

Takeaway. By hand, the full spectrum = ladders of eigenvalues set by the Jacobian at each fixed point.

System and fixed points ($f'(x) = 1 - 3x^2$):

$$\begin{aligned}
\dot{x} &= f(x) \\
&= x - x^3 \\
&= x(1-x)(1+x), && \text{factor} \\
f'(0) &= +1 \text{ (unstable)}, && \text{evaluate the derivative } f' = 1 - 3x^2 \\
f'(\pm 1) &= -2 \text{ (stable)}.
\end{aligned}$$

- Local rates $\mu_0 = +1$, $\mu_\pm = -2$ set the whole spectrum.

Solve the eigenfunction ODE. In 1-D $\mathcal{L}\psi = f(x)\psi'$, so $\mathcal{L}\psi = \lambda\psi$ is separable:

$$\begin{aligned}
&\frac{\partial \psi}{\partial x}(x - x^3) = \lambda \psi && \text{eigenfunction equation for the generator} \\
&\qquad \Longrightarrow \frac{\partial \psi}{\psi} = \lambda \frac{\partial x}{x(1-x)(1+x)}. && \text{separate variables}
\end{aligned} \tag{1}$$

Partial fractions $\dfrac{1}{x(1-x)(1+x)} = \dfrac{A}{x} + \dfrac{B}{1-x} + \dfrac{C}{1+x}$; from $1 = A(1-x)(1+x) + Bx(1+x) + Cx(1-x)$:

$$\begin{aligned}
x = 0 &: A = 1, \\
x = 1 &: B = \tfrac{1}{2}, \\
x = -1 &: C = -\tfrac{1}{2}.
\end{aligned}$$

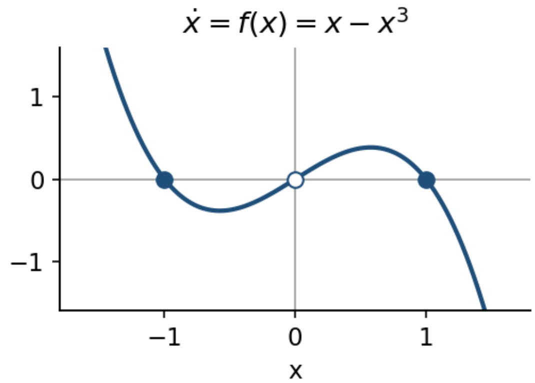

Figure 1: Vector field $f(x) = x - x^3$; filled = stable (± 1), open = unstable (0).

Integrate term by term (merging half-logs via $\ln|1-x| + \ln|1+x| = \ln|1-x^2|$):

$$
\begin{aligned}
\ln|\psi| &= \lambda\Big[\ln|x| - \tfrac{1}{2}\ln|1-x^2|\Big] && \text{integrate the partial fractions} \\
&= \lambda \ln \frac{|x|}{\sqrt{|1-x^2|}} && \text{combine the logs} \\
\implies \quad & \boxed{\psi_\lambda(x) = \left(\tfrac{|x|}{\sqrt{|1-x^2|}}\right)^{\lambda}} && \text{exponentiate}
\end{aligned}
$$

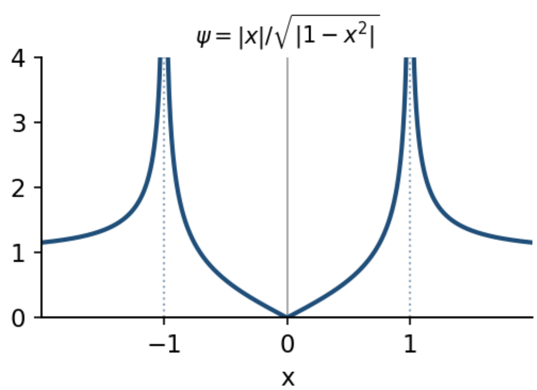

Figure 2: The magnitude $|x|/\sqrt{|1-x^2|}$ for $\lambda = 1$: cusp at 0, walls at ± 1.

- One formula for *every* λ (eigenfunctions defined up to scale).
- Smooth only on the four intervals $(-\infty, -1), (-1, 0), (0, 1), (1, \infty)$ (where $f \neq 0$).
- Sign hidden in the abs value: $|x| = \text{sign}(x) \cdot x$, and $\text{sign}(x)$ is a $\lambda = 0$ eigenfunction, so $\text{sign}(x)\,\psi_\lambda$ is also a λ-eigenfunction.

Quantize λ by demanding smoothness at a fixed point.

- **At $x = 0$:** $\psi_\lambda \approx x^\lambda$, smooth $\Leftrightarrow \lambda \in \{0, 1, 2, \dots\}$. Principal $\lambda = 1$:

$$\begin{aligned}
\psi(x) &= \frac{x}{\sqrt{1-x^2}} && \text{principal eigenfunction } (\lambda = 1)\\
\psi' &= (1-x^2)^{-3/2} \\
f \cdot \psi' &= x(1-x^2)^{-1/2} \\
&= \psi \quad (\lambda = 1 = \mu_0). && \text{confirms it is an eigenfunction } (\lambda = 1)
\end{aligned}$$

- **At $x = \pm 1$:** $\psi_\lambda \sim (1-x^2)^{-\lambda/2}$, finite $\Leftrightarrow \lambda \in \{0, -2, -4, \dots\}$. Principal $\lambda = -2$:

$$\begin{aligned}
\psi(x) &= \frac{1-x^2}{x^2} && \text{principal eigenfunction } (\lambda = -2)\\
&= \frac{1}{x^2} - 1 \\
f \cdot \psi' &= (x - x^3)(-2x^{-3}) && \text{differentiate, then multiply by } f\\
&= -2\psi \quad (\lambda = -2 = \mu_\pm). && \text{confirms it is an eigenfunction } (\lambda = -2)
\end{aligned}$$

- Ladder spacing = Jacobian $\mu = f'(x^*)$: $n\mu_0 = n$ at 0; $n\mu_\pm = -2n$ at ± 1, $n = 0, 1, 2, \dots$

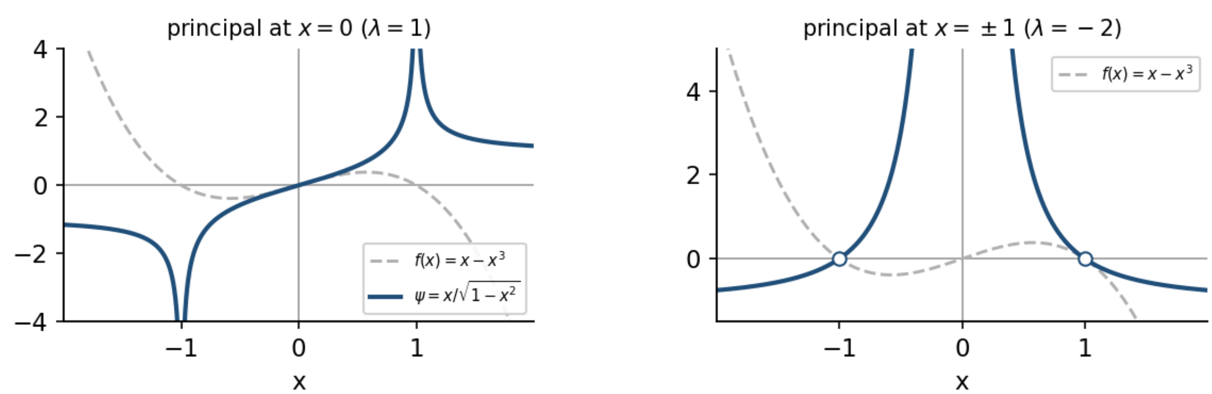

Figure 3: Principal eigenfunctions. Left: unstable point, $\psi = x/\sqrt{1-x^2}$ ($\lambda = +1$). Right: stable points, $\psi = (1-x^2)/x^2$ ($\lambda = -2$).

Observations.

- Stable eigenfunctions ($\lambda = -2$) vanish at ± 1 (general rule: $\operatorname{Re}\lambda < 0 \Rightarrow$ vanish on the attractor).
- Unstable eigenfunction ($\lambda = +1$) is 0 at $x = 0$, unbounded at ± 1.
- Separatrix (basin boundary) = where stable eigenfunctions blow up / the unstable one $\to 0$.
- **Tangency to f.** Near each fixed point one eigenfunction is tangent to f; matching the sign of the slope selects the branch. At $x = -1$ that branch is

$$\operatorname{sign}(x)\,\frac{1-x^2}{x^2} \qquad (\text{still } \lambda = -2 : \ \operatorname{sign}(x)\ [\lambda = 0]\ \times\ \text{principal } [\lambda = -2]).$$

This is how Koopman eigenspectra *globally* extend Hartman–Grobman (local Jacobian linearization) across the whole basin (Lan & Mezić 2013).

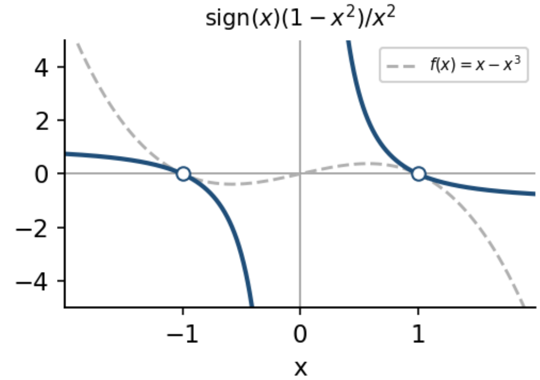

Figure 4: The signed branch $\mathrm{sign}(x)(1-x^2)/x^2$, tangent to f (dashed) near $x=-1$ — the $\lambda=-2$ eigenfunction that linearizes the basin, not just the fixed point.

The $\lambda = 0$ eigenspace = basin indicators

Takeaway. Eigenvalue-1 ($\lambda = 0$) eigenfunctions label which attractor you fall into.

- Set $\lambda = 0$ in (1): $f(x)\,\psi'(x) = 0$. Since $f \neq 0$ off the fixed points, ψ is constant on each invariant interval:

$$\psi(x) = C_1 \mathbf{1}_{x<-1} + C_2 \mathbf{1}_{x\in[-1,0)} + C_3 \mathbf{1}_{x\in[0,1)} + C_4 \mathbf{1}_{x\geq 1} \quad \text{(4-D eigenspace).}$$

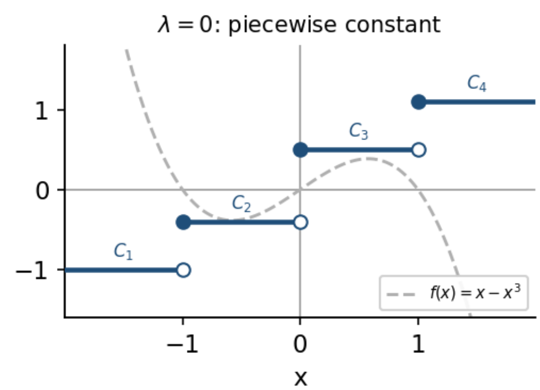

Figure 5: A general $\lambda = 0$ eigenfunction: constant on each interval; $C_1, \ldots, C_4$ free.

- **Basin indicators** (physical subclass, $C_1 = C_2$, $C_3 = C_4$, jump on the separatrix $x = 0$):

$$\mathbf{1}_{B_{-1}} = \mathbf{1}_{\{x<0\}},$$
$$\mathbf{1}_{B_{+1}} = \mathbf{1}_{\{x>0\}}.$$

Lemma 1 (Basin indicators are unit-eigenvalue eigenfunctions)**.** *For an autonomous flow with disjoint basins $B_1, \ldots, B_m$, $U\mathbf{1}_{B_i} = \mathbf{1}_{B_i}$ in $L^2(\mu)$; the $\mathbf{1}_{B_i}$ are mutually orthogonal, so $\lambda = 1$ has multiplicity $\geq m$.*

Proof. B_i forward-invariant: $x \in B_i \Rightarrow g(x) \in B_i$, so $\mathbf{1}_{B_i}(g(x)) = \mathbf{1}_{B_i}(x)$, i.e. $U\mathbf{1}_{B_i} = \mathbf{1}_{B_i}$. Orthogonality is immediate from disjointness. □

Stability

Takeaway. Stable eigenfunctions pin the attractor to their zero set — and hand you a Lyapunov function for free.

- $\operatorname{Re}\lambda < 0 \Rightarrow \psi(x(t)) \to 0$ along every trajectory, so (continuity) ψ vanishes on the attractor $\mathcal{A}$:

$$\operatorname{Re}\lambda < 0 \iff \psi|_{\mathcal{A}} = 0,$$
$$\mathcal{A} = \bigcap_i \{x : \psi_i(x) = 0\} \quad \text{(equilibria and limit cycles alike).}$$

- Lyapunov function (nonnegative, $= 0$ on $\mathcal{A}$): $V(x) = \left(\sum_i |\psi_i(x)|^p\right)^{1/p}$ (or $\max_i |\psi_i|$).
- Contracts at the slowest stable rate $\operatorname{Re}\lambda_1 = \max_i \operatorname{Re}\lambda_i < 0$:

$$\begin{aligned} V(x(t))^p &= \sum_i e^{p \operatorname{Re}\lambda_i t} |\psi_i(x(0))|^p && \text{each eigenfunction is a single exponential} \\ &\le e^{p \operatorname{Re}\lambda_1 t} \sum_i |\psi_i(x(0))|^p && \text{bound by the slowest rate} \\ &= e^{p \operatorname{Re}\lambda_1 t} V(x(0))^p, && \text{definition of } V \end{aligned}$$

so $V(x(t)) \le e^{\operatorname{Re}\lambda_1 t} V(x(0))$.

Conjugacy

Takeaway. Conjugate systems share a Koopman spectrum

Theorem 2 (Spectral equivalence of conjugate systems (Budišić et al., 2012))**.** *Let $S : M \to M$, $T : N \to N$ be conjugate via a homeomorphism $h : N \to M$ with $S \circ h = h \circ T$. If $U_S\psi = \lambda\psi$, then $\psi \circ h$ is an eigenfunction of U_T with the same λ.*

Proof. With $x = h(y)$:

$$\begin{aligned} U_S\psi(x) &= \psi\big(S(x)\big) && \text{definition of the Koopman operator} \\ &= \psi \circ (S \circ h)(y) = \psi \circ (h \circ T)(y) = U_T(\psi \circ h) && \text{apply the conjugacy relationship} \\ &= \lambda\psi(x) = \lambda(\psi \circ h)(y) \\ &\Rightarrow\ U_T(\psi \circ h) = \lambda(\psi \circ h) && \text{eigenvalue definition and apply conjugacys} \end{aligned}$$

□

- Graded "how close to conjugate": $d(S, T) = \min_h \|T - h^{-1} \circ S \circ h\|$.
- Lift $\Rightarrow$ similarity $U_T = C^{-1}U_S C$ (with $C\psi = \psi \circ h$): the nonlinear h becomes a linear change of basis, so compare *spectra* (basis-invariant).

Conservation laws

Takeaway. Conserved quantities are exactly the eigenvalue-1 eigenfunctions.

- $H(g(x)) = H(x)$ is the eigenvalue equation: $UH = H \circ g = H = 1 \cdot H$ ($\lambda = 1$). Energy, momentum, symmetry invariants.
- Harmonic oscillator $\dot{q} = p$, $\dot{p} = -\omega^2 q$: the complex coordinate $\psi = \omega q + ip$ is a Koopman eigenfunction. Since $\mathcal{L}\psi = \dot{\psi}$ along trajectories,

$$\begin{aligned}
\dot{\psi} &= \omega\dot{q} + i\dot{p} && \text{differentiate the coordinate} \\
&= \omega p - i\omega^2 q && \text{substitute the dynamics} \\
&= -i\omega(\omega q + ip) && \text{factor out } -i\omega \\
&= -i\omega\,\psi, && \text{recognize } \psi
\end{aligned}$$

  so $\mathcal{L}\psi = -i\omega\psi$ ($\lambda = -i\omega$), and the conjugate $\bar{\psi} = \omega q - ip$ gives $\lambda = +i\omega$. Products then follow by the additive lattice: $\mathcal{L}|\psi|^2 = \dot{\psi}\,\bar{\psi} + \psi\,\dot{\bar{\psi}} = (-i\omega + i\omega)|\psi|^2 = 0$, consistent with $|\psi|^2 = \omega^2 q^2 + p^2 = 2H$ being conserved.

6 Invariant & Almost-Invariant Sets

Takeaway. Perron–Frobenius eigenvalues just below 1 reveal metastable sets — the near-invariant regions where the system lingers.

- Invariant density: $\mathcal{P}\rho = \rho$ ($\lambda = 1$) — generalizes basins & conservation laws to stochastic/exotic sets.
- Almost-invariance of A: $\varrho(A) = \Pr[x_{t+1} \in A \mid x_t \in A] = 1 - \epsilon$.
- An eigenfunction $\mathcal{P}\psi = \lambda\psi$ with $\lambda \lesssim 1$ splits state space by sign into two almost-invariant sets, $A_+ = \{\psi > 0\}$ and $A_- = \{\psi < 0\}$. (Here ψ is the eigenfunction in $L^2(\mu)$, normalized so $\langle\psi,\psi\rangle_\mu = 1$ and mean-zero, $\mathbb{E}_\mu[\psi] = 0$.) Idealize $\psi = +1$ on A_+ and $\psi = -1$ on A_-, and let $s = \Pr[\text{stay in the same set for one step}]$.
- Watch the product $\psi(x_t)\,\psi(x_{t+1})$ along a trajectory: $+1$ if the state stays put (same sign), -1 if it switches. Averaging over the stationary dynamics:

$$\begin{aligned}
\mathbb{E}\big[\psi(x_t)\,\psi(x_{t+1})\big] &= (+1)\,s + (-1)(1-s) && +1 \text{ stay, } -1 \text{ switch} \\
&= 2s - 1. && \text{simplify}
\end{aligned}$$

- That average *is* the eigenvalue, since ψ is a normalized eigenfunction of $\mathcal{P}$:

$$\mathbb{E}\big[\psi(x_t)\,\psi(x_{t+1})\big] = \langle\psi, \mathcal{P}\psi\rangle_\mu = \lambda\,\langle\psi,\psi\rangle_\mu = \lambda \qquad \langle\psi,\psi\rangle_\mu = 1 \text{ (normalized)}$$

- Equate: $\lambda = 2s - 1 \Rightarrow s = \frac{1}{2}(1 + \lambda)$. So λ near 1 means it almost always stays (escape time $\sim (1-\lambda)^{-1}$); $\lambda = 1$ is exactly invariant.
- m metastable regions $\Rightarrow$ m eigenvalues near 1:

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \gg \lambda_{m+1}.$$

- Use $(\psi_1(x), \ldots, \psi_m(x))$ as coordinates $\Rightarrow$ the m sets appear as m clusters: spectral clustering with $\mathcal{P}$ in place of a graph Laplacian. Gap after λ_m counts the sets.

7 DSA: ring vs. cut ring

Takeaway. Same equations, one different boundary condition — and the spectrum cleanly separates the ring (traveling waves, degenerate) from the cut ring (standing waves, simple).

Both models (radius relaxes, phase diffuses); differ *only* in the θ boundary condition:

$$\begin{aligned}
dr &= -\mu(r-r^*)\,dt + \sigma_r\,dW_r,\\
d\theta &= \sigma_\theta\,dW_\theta,\\
\text{b.c.:}\ &\begin{cases}\text{ring: periodic } \theta \sim \theta + L,\\ \text{cut ring: reflecting } \partial_\theta\psi|_0 = \partial_\theta\psi|_L = 0 \text{ (Neumann).}\end{cases}
\end{aligned}$$

Stochastic generator (Itô; splits into radial + angular):

$$\begin{aligned}
\mathcal{L}\psi &= \lim_{dt\to 0}\frac{\mathbb{E}[d\psi]}{dt} && \text{stochastic (Itô) generator}\\
&= \underbrace{-\mu(r-r^*)\partial_r + \tfrac{\sigma_r^2}{2}\partial_r^2}_{\mathcal{L}_r}\,\psi + \underbrace{\tfrac{\sigma_\theta^2}{2}\partial_\theta^2}_{\mathcal{L}_\theta}\,\psi.
\end{aligned}$$

- Separate $\psi(r,\theta) = R(r)\,g(\theta)$, so $\lambda = \lambda_r + \lambda_\theta$. Radial part shared (Ornstein–Uhlenbeck): $\lambda_r^{(k)} = -k\mu$, $k = 0, 1, 2, \dots$ (Hermite). All the difference is angular: $\frac{\sigma_\theta^2}{2} g'' = \lambda_\theta g$.
- **Ring (periodic).** $e^{ikL} = 1 \Rightarrow k_n = 2\pi n/L$, traveling waves $g_n = e^{i2\pi n\theta/L}$:

$$\begin{aligned}
\lambda_\theta^{(n)} &= \tfrac{\sigma_\theta^2}{2}\big(i\tfrac{2\pi n}{L}\big)^2 && \text{apply the generator}\\
&= -\frac{2\pi^2\sigma_\theta^2 n^2}{L^2}, \quad n \in \mathbb{Z} && \text{simplify}
\end{aligned}$$

($n \geq 1$ **doubly degenerate**: CW/CCW, S^1 symmetry).

- **Cut ring (Neumann).** Standing waves (even) $g_n = \cos(n\pi\theta/L)$, $k_n = n\pi/L$:

$$\lambda_\theta^{(n)} = -\frac{\sigma_\theta^2}{2}\big(\tfrac{n\pi}{L}\big)^2, \quad n = 0, 1, 2, \dots \qquad \text{apply the generator to the standing wave}$$

(**simple** : the cut breaks the CW/CCW pairing).

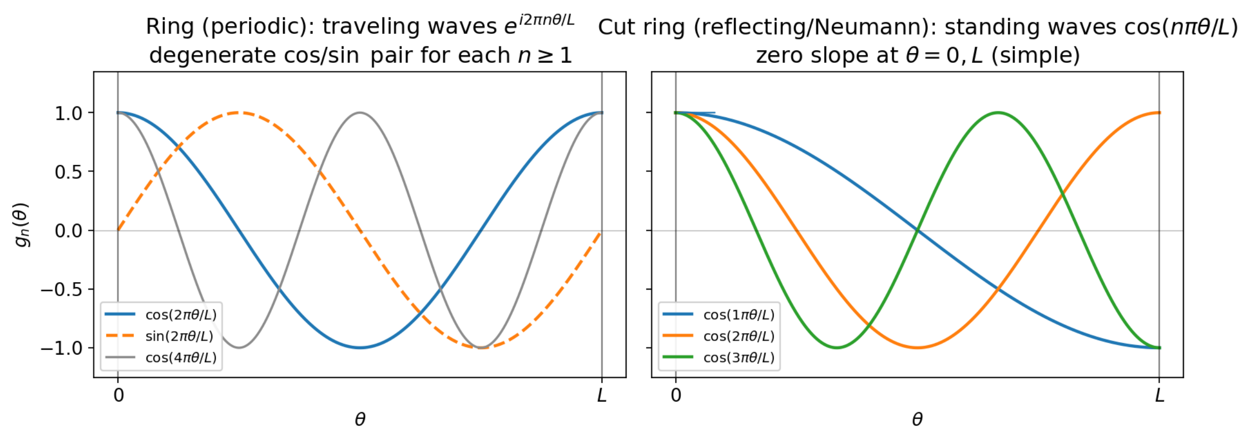

Figure 6: Angular eigenfunctions. Left, ring: traveling waves (degenerate cos / sin pairs). Right, cut ring: cosine standing waves (simple).